

COMMUTING MAPS ON RANK- k MATRICES

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ABSTRACT. Let $n \geq 2$ be a natural number. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over a field \mathbb{K} . Fix natural number k satisfying $1 < k \leq n$. Under a mild technical assumption over \mathbb{K} we will show that additive maps $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ such that $[G(x), x] = 0$ for every rank- k matrix $x \in M_n(\mathbb{K})$ are of form $\lambda x + \mu(x)$, where $\lambda \in Z$, $\mu : M_n(\mathbb{K}) \rightarrow Z$, and Z stands for the center of $M_n(\mathbb{K})$. Furthermore, we shall see an example that there are additive maps such that $[G(x), x] = 0$ for all rank-1 matrices that are not of the form $\lambda x + \mu(x)$. We will also discuss the m -additive case.

Let $n \geq 2$ be a natural number, and let \mathbb{K} be a field. An additive map $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ is called commuting if for each $x \in M_n(\mathbb{K})$ the equality $[G(x), x] = G(x)x - xG(x) = 0$ holds. In [4] we proved that if $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ is an additive map and $[G(x), x] = 0$ for every invertible $x \in M_n(\mathbb{K})$ then G has the form

$$G(x) = \lambda x + \mu(x), \quad (1)$$

where $\lambda \in Z$, $\mu : M_n(\mathbb{K}) \rightarrow Z$ is an additive map, $Z = \mathbb{K} \cdot I$ is the center of $M_n(\mathbb{K})$, I is the identity matrix, and \mathbb{K} is any field but \mathbb{Z}_2 . Thus, writing this result in terms of rank, we could say that: If $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ is an additive map such that $[G(x), x] = 0$ for each rank- n matrix $x \in M_n(\mathbb{K})$ then G has the form (1). So, it gave rise to a natural question: For a fixed natural number k satisfying $1 \leq k \leq n-1$ does an additive map G from $M_n(\mathbb{K})$ to itself with the property that $[G(x), x] = 0$ for every rank- k matrix $x \in M_n(\mathbb{K})$ have the description (1)? Under the assumption that \mathbb{K} is a field that either $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} > 3$ the answer will be yes when $1 < k \leq n-1$. In the case that $k = 1$ we have a negative answer for $n \geq 3$.

Now, let us see how this problem can be formulated in terms of m -additive maps. Let $n > m > 1$ be natural numbers, and let \mathbb{K} be a field such that $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} \geq m+1$. We say that $G : M_n(\mathbb{K})^m \rightarrow M_n(\mathbb{K})$ is m -additive if G is additive in each component, that is, $G(x_1, \dots, x_i + y_i, \dots, x_m) = G(x_1, \dots, x_i, \dots, x_m) + G(x_1, \dots, y_i, \dots, x_m)$ for all $x_i, y_i \in M_n(\mathbb{K})$, and $i \in \{1, \dots, m\}$. The map $T : M_n(\mathbb{K}) \rightarrow$

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$M_n(\mathbb{K})$ defined by $T(x) = G(x, \dots, x)$ is known as the trace of G , and such traces are called commuting if for each $x \in M_n(\mathbb{K})$ the equality $[G(x, \dots, x), x] = 0$ holds.

In the present work, we will characterize all m -additive maps $G : M_n(\mathbb{K})^m \rightarrow M_n(\mathbb{K})$ such that $[G(x, \dots, x), x] = 0$ for every rank- k matrix $x \in M_n(\mathbb{K})$, where k is a fixed natural number satisfying $(m+1) \leq k \leq n$.

We just want to remind that any (m) -additive map $G : M_n(\mathbb{K})^{(m)} \rightarrow M_n(\mathbb{K})$ is (m) -linear over \mathbb{Q} (respectively \mathbb{Z}_q) when $\text{char } \mathbb{K} = 0$ (respectively $\text{char } \mathbb{K} = q$). This fact will be largely used in this paper. Furthermore, it should be mentioned that the set of all nonzero elements of \mathbb{K} will be denoted by \mathbb{K}^* .

Let us start with the additive case. First, we need a couple of auxiliary results.

Lemma 1. *Let $n \geq 3$ be a natural number, and let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} . Fix a natural number k satisfying $1 < k \leq n$, and numbers $i, j \in \{1, \dots, n\}$. Then there exists a matrix $B = B(i, j) \in M_n(\mathbb{K})$ such that $ze_{ij} + tB$ has rank k for all nonzero $z, t \in \mathbb{K}$.*

Proof. Fix i, j, k as described in the statement. For each $v \in \{2, \dots, k\}$ choose $i_v, j_v \in \{1, \dots, n\}$ such that $i \neq i_u, j \neq j_u, i_u \neq i_v$, and $j_u \neq j_v$ when $u \neq v$ ($u, v \in \{2, \dots, k\}$). Set $B = \sum_{v=2}^k e_{i_v j_v}$. Clearly $ze_{ij} + tB$ has rank k for all $z, t \in \mathbb{K}^*$. \square

In the same way we can prove the following:

Lemma 2. *Let $n \geq 3$ be a natural number, and let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} . Fix a natural number k satisfying $1 < k \leq n$, and numbers $i, j, k, l \in \{1, \dots, n\}$, where $(i, j) \neq (k, l)$. Then there exists a matrix $B = B(i, j, k, l) \in M_n(\mathbb{K})$ such that $ze_{ij} + we_{kl} + tB$ has rank k for all nonzero $z, w, t \in \mathbb{K}$.*

Theorem 3. *Let $n \geq 3$ be a natural number. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} with center $Z = \mathbb{K} \cdot I$, where I is the identity matrix. Fix a natural number k satisfying $1 < k \leq n - 1$. Assume that $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} > 3$. Let $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ be an additive map such that*

$$[G(x), x] = 0 \quad \text{for every rank-}k \text{ matrix } x \in M_n(\mathbb{K}). \quad (2)$$

Then there exist an element $\lambda \in Z$ and an additive map $\mu : M_n(\mathbb{K}) \rightarrow Z$ such that

$$G(x) = \lambda x + \mu(x) \quad \text{for each } x \in M_n(\mathbb{K}). \quad (3)$$

Proof. Let $z, w \in \mathbb{K}$, and $i, j, k, l \in \{1, \dots, n\}$, where $(i, j) \neq (k, l)$. All the identities that will be obtained during this proof can be easily checked when either z or w are zero. So, we may assume that z, w are nonzero elements of \mathbb{K} . Using Lemma 1, we

can find a matrix $B \in M_n(\mathbb{K})$ such that $ze_{ij} + tB$ has rank k for all nonzero $t \in \mathbb{K}$. Then equation (2) tells us that $[G(ze_{ij} + tB), ze_{ij} + tB] = 0$ for all $t \in \mathbb{K}^*$. Replacing t by 1, -1 and 2 we arrive at (because $\text{char } \mathbb{K} > 3$):

$$[G(ze_{ij}), ze_{ij}] = 0 \quad \text{for all } z \in \mathbb{K}, \quad \text{and } i, j \in \{1, \dots, n\}. \quad (4)$$

On the other hand, Lemma 2 guarantees the existence of an element $D \in M_n(\mathbb{K})$ such that $ze_{ij} + we_{kl} + tD = c + tD$ has rank k . Again, equation (2) provides $[G(c + tD), c + tD] = 0$. So, it follows from the same argument that we used in the proof of equation (4) that $[G(c), c] = 0$, which becomes

$$[G(ze_{ij}), we_{kl}] + [G(we_{kl}), ze_{ij}] = 0, \quad (5)$$

since $c = ze_{ij} + we_{kl}$ (note that equation (4) has been used).

Hence, $[G(ze_{ij}), we_{kl}] + [G(we_{kl}), ze_{ij}] = 0$, for all $i, j, k, l \in \{1, \dots, n\}$, and $z, w \in \mathbb{K}$. Thus, $[G(x), x] = 0$ for each $x \in M_n(\mathbb{K})$, because G is additive and $M_n(\mathbb{K})$ is additively generated by ze_{ij} . The desired result now is obtained from the well-known theorem on commuting maps due to Brešar (see the original paper [1], or the survey paper [2, Corollary 3.3], or the book [3, Corollary 5.28]).

□

Now, we will study commuting maps on the set of rank-1 matrices. The proof of the next result can be found in [4, Theorem 1] (case $n = 2$).

Theorem 4. *Let \mathbb{K} be a field, and $n = 2$. Let $M_2(\mathbb{K})$ be the ring of all 2×2 matrices over \mathbb{K} with center $Z = \mathbb{K} \cdot I$, where I is the identity matrix. Let $G : M_2(\mathbb{K}) \rightarrow M_2(\mathbb{K})$ be an additive map such that*

$$[G(x), x] = 0 \quad \text{for every rank-1 matrix } x \in M_2(\mathbb{K}). \quad (6)$$

Then there exist an element $\lambda \in Z$ and an additive map $\mu : M_2(\mathbb{K}) \rightarrow Z$ such that

$$G(x) = \lambda x + \mu(x) \quad \text{for each } x \in M_2(\mathbb{K}). \quad (7)$$

Surprisingly, Theorem 3 fails when $k = 1$ and $n \geq 3$.

Example 1. Let \mathbb{K} be any field. Fix a natural number $n \geq 3$. Let us define a linear map from $M_n(\mathbb{K})$ to itself in the following way: $G(ze_{11}) = -ze_{n2}$, $G(ze_{1n}) = ze_{12}$, $G(ze_{21}) = ze_{n1}$, $G(ze_{2n}) = \sum_{j=2}^n ze_{jj}$, and $G(ze_{ij}) = 0$ otherwise, where $z \in \mathbb{K}$. The

map G is commuting on the set of rank-1 matrices. Indeed, let x be a rank-1 matrix. Thus, x can be written in the following way:

$$x = \sum_{j=1}^n \lambda_j \sum_{i=1}^n x_i e_{ij},$$

where $x_v, \lambda_v \in \mathbb{K}$ for all $v \in \{1, \dots, n\}$, and $\lambda_l = 1$ for some $l \in \{1, \dots, n\}$. Therefore, for the above x , we have:

$$G(x) = \left(-\lambda_1 x_1 e_{n2} + \lambda_n x_1 e_{12} + \lambda_1 x_2 e_{n1} + \sum_{j=2}^n \lambda_n x_2 e_{jj} \right).$$

By standard computations, we see that:

$$G(x)x = \lambda_n x_2 \left[\sum_{j=1}^n \lambda_j \sum_{i=1}^n x_i e_{ij} \right] = xG(x).$$

Hence, $[G(x), x] = 0$ for all rank-1 matrix $x \in M_n(\mathbb{K})$, but G is not of the form (1) as it maps the identity element I to a noncentral element $-e_{n2}$.

Now, we will investigate commuting traces of m -additive maps.

First of all, observe that we may assume that our m -additive map $G : M_n(\mathbb{K})^m \rightarrow M_n(\mathbb{K})$, whose trace commute on the set of rank- k matrices ($m+1 \leq k \leq n$), is symmetric because we can replace G with $\sum_{\sigma \in S_m} G(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = G'(x_1, \dots, x_m)$, and it is clear that G' is symmetric and that $[G'(x, \dots, x), x] = 0$ for each rank- k matrix $x \in M_n(\mathbb{K})$. Notice that $G'(x, \dots, x) = m!G(x, \dots, x)$, thus the trace of G' is identically zero if and only if the trace of G is, since either $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} \geq m+1$.

Secondly, we will state the following result that is a mere generalization of the Lemma 2.

Lemma 5. *Let \mathbb{K} be a field, and let m, n be natural numbers, where $n \geq m+1$, and $m > 1$. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} . Fix a natural number k satisfying $(m+1) \leq k \leq n$. Let v be a natural number such that $1 \leq v \leq m+1$. For each $r \in \{1, \dots, v\}$, let $i_r, j_r \in \{1, \dots, n\}$, where $(i_u, j_u) \neq (i_v, j_v)$ when $u \neq v$. Then there exists a matrix $B = B((i_r, j_r) | r \in \{1, \dots, v\}) \in M_n(\mathbb{K})$ such that $\sum_{r=1}^v z_r e_{i_r j_r} + tB$ has rank k for all nonzero $z_1, \dots, z_v, t \in \mathbb{K}$.*

The argument needed in the proof of the next Theorem is similar to the one used in [5, Theorem 2], but for the completeness sake we will provide all the details.

Theorem 6. *Let \mathbb{K} be a field, and let m, n be natural numbers, where $n \geq m + 1$, and $m > 1$. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} with center $Z = \mathbb{K} \cdot I$, where I is the identity matrix. Fix a natural number k satisfying $(m + 1) \leq k \leq n$. Let $G : M_n(\mathbb{K})^m \rightarrow M_n(\mathbb{K})$ be a symmetric m -additive map such that*

$$[G(x, \dots, x), x] = 0 \quad \text{for every rank-}k \text{ matrix } x \in M_n(\mathbb{K}). \quad (8)$$

Assume that $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} > m + 1$. Assume also that \mathbb{K} contains at least $m + 4$ elements. Then, there exist $\mu_0 \in Z$ and maps $\mu_i : M_n(\mathbb{K}) \rightarrow Z$, $i \in \{1, \dots, m\}$, such that each μ_i is the trace of an i -additive map and $G(x, \dots, x) = \mu_0 x^m + \mu_1(x) x^{m-1} + \dots + \mu_{m-1}(x) x + \mu_m(x)$ for each $x \in M_n(\mathbb{K})$.

Proof. For each $r \in \{1, \dots, m + 1\}$, let $z_r \in \mathbb{K}^*$ and let $i_r, j_r \in \{1, \dots, n\}$, where $(i_u, j_u) \neq (i_v, j_v)$ when $u \neq v$. By Lemma 5 there exists a matrix $B \in M_n(\mathbb{K})$ such that $\sum_{r=1}^{m+1} z_r e_{i_r, j_r} + tB = c + tB$ has rank k for all nonzero $t \in \mathbb{K}$. It follows from (8) that $[G(c + tB, \dots, c + tB), c + tB] = 0$ for all $t \in \mathbb{K}^*$. Hence, $[G(c + tB, \dots, c + tB), c + tB] + [G(c + sB, \dots, c + sB), c + sB] = 0$ for all $s, t \in \mathbb{K}^*$. Letting $0 \neq s = -t$, and using the symmetricity of G , we get that:

$$\sum_{h=0}^{\zeta} \binom{m}{2h} [G(\underbrace{tB, \dots, tB}_{2h}, c, \dots, c), c] + \sum_{h=0}^{\varepsilon} \binom{m}{2h+1} [G(\underbrace{tB, \dots, tB}_{2h+1}, c, \dots, c), tB] = 0, \quad (9)$$

where $\zeta = \frac{m}{2}$, and $\varepsilon = \zeta - 1$ when m is even and $\zeta = \varepsilon = \frac{m-1}{2}$ when m is odd. For convenience let us set:

$$\alpha(h) = \binom{m}{2h} [G(\underbrace{B, \dots, B}_{2h}, c, \dots, c), c], \quad \text{where } h \in \{0, \dots, \zeta\}.$$

$$\gamma(h) = \binom{m}{2h+1} [G(\underbrace{B, \dots, B}_{2h+1}, c, \dots, c), B], \quad \text{where } h \in \{0, \dots, \varepsilon\}.$$

Observe that for each $t \in \{1, \dots, \varepsilon + 2\}$ we have obtained an equation of the form (9). It means that we got $(\varepsilon + 2)$ distinct equations of such form (this is true because $\text{char } \mathbb{K} > m + 1$, and $|\mathbb{K}| \geq m + 4$). Thus, using matrix notation we have the

following:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2^2 & 2^4 & \dots & 2^a \\ 1 & 3^2 & 3^4 & \dots & 3^a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (\varepsilon+2)^2 & (\varepsilon+2)^4 & \dots & (\varepsilon+2)^a \end{pmatrix} \begin{pmatrix} \alpha(0) \\ \alpha(1) + \gamma(0) \\ \alpha(2) + \gamma(1) \\ \alpha(3) + \gamma(2) \\ \vdots \\ y \end{pmatrix} = 0,$$

where $a = m$, (respectively $a = m + 1$) and $y = \alpha(\zeta) + \gamma(\varepsilon)$ (respectively $y = \gamma(\varepsilon)$) when m is even (respectively m is odd).

Because the determinant of the Vandermonde matrix formed by the coefficients of the system is not zero, we get that $\alpha(0) = [G(c, \dots, c), c] = 0$, and this implies that:

$$\left[G\left(\sum_{r=1}^{m+1} z_r e_{i_r j_r}, \dots, \sum_{r=1}^{m+1} z_r e_{i_r j_r}\right), \sum_{r=1}^{m+1} z_r e_{i_r j_r} \right] = 0, \quad (10)$$

since $c = \sum_{r=1}^{m+1} z_r e_{i_r j_r}$.

Now, fix $v \in \{1, \dots, m\}$. Notice that the previous argument works if the element $c = \sum_{r=1}^{m+1} z_r e_{i_r j_r}$ is replaced by $c_v = \sum_{r=1}^v z_r e_{i_r j_r}$, where $z_r \in \mathbb{K}^*$, $i_r, j_r \in \{1, \dots, n\}$ for each $r \in \{1, \dots, v\}$ with $(i_u, j_u) \neq (i_v, j_v)$ when $u \neq v$. Thus, in the same fashion we can prove that $[G(c_v, \dots, c_v), c_v] = 0$, which yields:

$$\left[G\left(\sum_{r=1}^v z_r e_{i_r j_r}, \dots, \sum_{r=1}^v z_r e_{i_r j_r}\right), \sum_{r=1}^v z_r e_{i_r j_r} \right] = 0, \quad \text{for all } v \in \{1, \dots, m\}. \quad (11)$$

So, after using that G is m -additive and the identity (11) (for each $v \in \{1, \dots, m\}$), the equality (10) becomes:

$$\sum_{\sigma \in \mathcal{S}_{m+1}} [G(z_{i_{\sigma(1)}} e_{i_{\sigma(1)} j_{\sigma(1)}}, \dots, z_{i_{\sigma(m)}} e_{i_{\sigma(m)} j_{\sigma(m)}}), z_{i_{\sigma(m+1)}} e_{i_{\sigma(m+1)} j_{\sigma(m+1)}}] = 0. \quad (12)$$

Note that the above equation holds trivially when $z_r = 0$ for some $r \in \{1, \dots, m+1\}$. Hence, writing any matrix $x \in M_n(\mathbb{K})$ as the sum of its entries $x_{ij} e_{ij}$ we can conclude from (12) that $[G(x, \dots, x), x] = 0$ for each $x \in M_n(\mathbb{K})$, since G is m -additive. The desired result now follows from [6, Theorem 3.1]. \square

As a consequence of the Theorem 6 we can easily obtain the following result that was proved in [5].

Corollary 7. *Let \mathbb{K} be a field, and let m, n be natural numbers, where $n \geq m + 1$, and $m > 1$. Let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over \mathbb{K} with center $Z = \mathbb{K} \cdot I$, where I is the identity matrix. Let $G : M_n(\mathbb{K})^m \rightarrow M_n(\mathbb{K})$ be a symmetric m -additive map such that*

$$[G(x, \dots, x), x] = 0, \quad \text{for every invertible (singular) } x \in M_n(\mathbb{K}). \quad (13)$$

Assume that $\text{char } \mathbb{K} = 0$ or $\text{char } \mathbb{K} > m + 1$. Assume also that \mathbb{K} contains at least $m + 4$ elements. Then, there exist $\mu_0 \in Z$ and maps $\mu_i : M_n(\mathbb{K}) \rightarrow Z$, $i \in \{1, \dots, m\}$, such that each μ_i is the trace of an i -additive map and $G(x, \dots, x) = \mu_0 x^m + \mu_1(x) x^{m-1} + \mu_{m-1}(x) x + \mu_m(x)$ for each $x \in M_n(\mathbb{K})$.

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